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2001 J. Phys. A: Math. Gen. 34 7633

(http://iopscience.iop.org/0305-4470/34/37/315)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **34** (2001) 7633–7638 | PII: S0305-4470(01)24486-2

# On multivariate $p$-adic $q$-integrals

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Received 19 April 2001, in final form 24 July 2001
Published 7 September 2001
Online at stacks.iop.org/JPhysA/34/7633

**Abstract**
By using multiple $p$-adic $q$-integrals, we define the $p$-adic $q$-$L$-function in $n$-variables and the $q$-extension of $p$-adic log multiple gamma functions. From these definitions, we show that the values of the $p$-adic $q$-$L$-function at positive integers can be expressed in terms of the $q$-extension of $p$-adic log multiple gamma functions.

PACS number: 02.10.De
Mathematics Subject Classification: 11S80

## 1. Introduction

Let $p$ be a fixed prime and let $\mathbb{C}_p$ denote the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. For $d$ a fixed positive integer with $(p, d) = 1$, let

$$X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z} \qquad X_1 = \mathbb{Z}_p$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}$$

where $a \in \mathbb{Z}$ lies in $0 \leqslant a < dp^N$.

The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. $C(\mathbb{Z}_p, \mathbb{C}_p)$ will denote the set of all continuous $f: \mathbb{Z}_p \to \mathbb{C}_p$. Let $U_1 \subset \mathbb{C}_p$ denote the open unit disc about 1 and $U_d = \{u \in \mathbb{C}_p \| u^d - 1|_p < 1\}$ the union of the open unit discs around $d$th root of unity. Let $U^m = U_d \times U_1^{m-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we

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normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leqslant 1$. Throughout this paper, we use the following notation:

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

In this paper, we construct the $p$-adic $q$-$L$-function in $n$-variables and the multiple $p$-adic $q$-log gamma function by using the values of the multiple $p$-adic $q$-integral. Finally, we give the formulae which express the values of the $p$-adic $q$-$L$-function in $n$-variables at positive integers in terms of the multiple $p$-adic $q$-log gamma functions.

In [1, 2], the authors studied orthogonal and symmetric operators in non-Archi-medean Hilbert spaces in connection with $p$-adic quantization.

Orthogonal isometric isomorphism of $p$-adic Hilbert spaces preserves precision of measurements. In [1], the authors also studied the properties of orthogonal operators.

As the quantum field theory allows infinite degree of freedom, we need to propose an infinite-dimensional non-Archimedean analysis if we wish to study quantum field theory with non-Archimedean valued fields. Such analysis has already been presented by Khrennikov in [3].

The quantization of a bosonic non-Archimedean valued field is carried out in the functional integral formalism [3]. Khrennikov [4] tried to build a $p$-adic picture of reality based on the field of $p$-adic numbers $\mathbb{Q}_p$ and the corresponding non-Archimedean analysis. He showed that many problems of description of reality with the aid of real numbers are induced by unlimited application of the non-Archimedean axiom. This axiom means that the physical observable can be measured with an infinite exactness. Khrennikov's $p$-adic model of physical reality is based on a finite exactness of measurement which violates the Archimedean axiom.

As with the above $p$-adic model of physical reality, our results stimulate quantum mechanics by using mathematical apparatus, namely, the properties of the $q$-analogue of zeta function, the definition of $p$-adic $q$-$L$-functions and $q$-Mahler's theory of $p$-adic $q$-integration with respect to a ring $\mathbb{Z}_p$ of $p$-adic integers. Iwasawa isomorphism and the $p$-adic $q$-log gamma functions are used in sections 2 and 3, repectively.

## 2. $p$-adic $q$-integral on compact subgroups of $\mathbb{C}_p$.

For $f \in C^{(1)}(\mathbb{Z}_p) = \{$the set of strictly differentiable functions on $\mathbb{Z}_p\}$, let us start with the expressions

$$\frac{1}{[p^N]} \sum_{0 \leqslant j < p^N} q^j f(j) = \sum_{0 \leqslant j < p^N} f(j)\mu_q\big(j + p^N\mathbb{Z}_p\big) \quad \text{(cf. [3, 5, 6])}$$

representing the $q$-analogue of the Riemann sums for $f$. The integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit ($N \to \infty$) of these sums, when it exists.

The $p$-adic $q$-integral of a function $f \in C^{(1)}(\mathbb{Z}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(x)\, \mathrm{d}\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{0 \leqslant j < p^N} f(j) q^j.$$

For $f \in C^{(1)}(\mathbb{Z}_p)$, it is easy to see that

$$\left| \int_{\mathbb{Z}_p} f(x)\, \mathrm{d}\mu_q(x) \right|_p \leqslant p||f||_1$$

where $||f||_1 = \sup\{|f(0)|_p, \sup_{x\neq y} |\frac{f(x)-f(y)}{x-y}|_p\}$.

If $f_n \to f$ in $C^{(1)}$, namely $||f_n - f||_1 \to 0$, then

$$\int_{\mathbb{Z}_p} f_n(x)\,\mathrm{d}\mu_q(x) \to \int_{\mathbb{Z}_p} f(x)\,\mathrm{d}\mu_q(x)$$

(see [4]). The $q$-analogue of the binomial coefficient is known as

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x][x-1]\cdots[x-n+1]}{[n]!} \qquad \text{where} \quad [n]! = \prod_{i=1}^{n}[i] \quad \text{(cf. [2]).}$$

Note that

$$\begin{bmatrix} x+1 \\ n \end{bmatrix} = \begin{bmatrix} x \\ n-1 \end{bmatrix} + q^x \begin{bmatrix} x \\ n \end{bmatrix} = q^{x-n} \begin{bmatrix} x \\ n-1 \end{bmatrix} + \begin{bmatrix} x \\ n \end{bmatrix}.$$

Thus we see that

$$\int_{\mathbb{Z}_p} \begin{bmatrix} x \\ n \end{bmatrix} \mathrm{d}\mu_q(x) = \frac{(-1)^n}{[n+1]} q^{n+1-\binom{n+1}{2}}.$$

For any $a = (a_1, a_2, \ldots, a_m) \in \mathbb{U}^m$, let $\mu_q$ denote the $p$-adic distribution on $X^m$ which is defined on the standard basis of the compact-open sets by

$$\mu_q(a + dp^N X^m) = \frac{q^a}{[dp^{N_1}][p^{N_2}]\cdots[p^{N_m}]}$$

where the notation $a + dp^N X^m = (a_1 + dp^{N_1}\mathbb{Z}_p) \times (a_2 + p^{N_2}\mathbb{Z}_p) \times \cdots \times (a_m + p^{N_m}\mathbb{Z}_p) \subset X^m$, $q^a$ denotes $\prod_j q^{a_j}$. Then we have the following:

**Theorem 1.** *For any uniformly differentiable function $f$: $X^m \to \mathbb{C}_p$, we have*

$$\int_{X^m} f(x)\,\mathrm{d}\mu_q(x)$$

*which is bounded and locally analytic in each $q$ on $U^m$.*

**Proof.** Theorem 1 is proved by the definition of $p$-adic $q$-integral [5, 6]. □

**Corollary 1.** *For any $b_1, b_2, \ldots, b_r \in \mathbb{Z}_{\geqslant 0}$, $y = (y_1, \ldots, y_r) \in \mathbb{Z}_p^r$, we see that*

$$\int_{\mathbb{Z}_p^r} [y_1]^{b_1}[y_2]^{b_2}\cdots[y_r]^{b_r}\,\mathrm{d}\mu_q(y)$$

*is the coefficient of $\frac{t_1^{b_1}\cdots t_r^{b_r}}{b_1!b_2!\cdots b_r!}$ in the Laurent expansion of the function*

$$\prod_{l=1}^{r}\left(\mathrm{e}^{\frac{t_l}{1-q}} \sum_{j=0}^{\infty} \frac{j+1}{[j+1]}(-1)^j \left(\frac{1}{1-q}\right)^j \frac{t_l^j}{j!}\right).$$

The proof of Corollary 1 is not difficult [5].

Note that

$$\int_{\mathbb{Z}_p^r} [y_1]^{b_1}\cdots[y_r]^{b_r}\,\mathrm{d}\mu_q(y) = \lim_{N_1,\ldots,N_r\to\infty} \frac{1}{\left[c_1 p^{N_1}\right]\cdots\left[c_r p^{N_r}\right]} \times \sum_{1\leqslant j\leqslant r}\ \sum_{0\leqslant m_j<c_j p^N} \left(q^{m_1}[m_1]^{b_1}\right)\cdots\left(q^{m_r}[m_r]^{b_r}\right)$$

where each $c_i \in \mathbb{N} = \{\text{the set of positive integers}\}$ for $(i = 1, 2, \ldots, r)$.

## 3. On the multiple *p*-adic *q*-log gamma functions

The one-variable $p$-adic $q$-log gamma functions were defined and their application has already been treated by Kim [5].

In this section, we construct the multiple $p$-adic $q$-log gamma functions and the $p$-adic $q$-$L$-functions in $n$-variables to give the formulae which express the values of the $p$-adic $q$-$L$-functions in $n$-variables at positive integers in terms of multiple $p$-adic $q$-log gamma functions.

For any positive integers $r$ and $n$, and $x_i \in \mathbb{C}_p^\times (\forall i)$, we define the $p$-adic $q$-$L$-functions in $n$-variables as follows:

$$L_{p,q}(s:r) = L_{p,q}(s_1, s_2, \ldots, s_n : r) = \int_{\mathbb{Z}_p^{\times r}} \prod_{1\leqslant i\leqslant n} \frac{[x_i + y_1 + \cdots + y_r]^{-s_i+1}}{s_i - 1} \, \mathrm{d}\mu_q(y).$$

In the case of $n = r = 1$, note that $L_{p,q}(s:r)$ is the same $p$-adic $q$-$L$-function, $L_{p,q}(s, \chi^0)$, which is defined in [1]. Let $k_1, k_2, \ldots, k_n$ be positive integers.

Indeed, we see that

$$\begin{aligned} L_{p,q}(1-k:r) &= L_{p,q}(1-k_1, 1-k_2, \ldots, 1-k_n : r) \\ &= (-1)^n \prod_{1\leqslant i\leqslant n} \frac{1}{k_i} \left( \beta_{k_i}^{(r)}(x_i, q) - [p]^{k_i - r} \beta_{k_i}^{(r)}(x_i, q^p) \right) \end{aligned}$$

where $\beta_{k_i}^{(r)}(x_i, q)$ are $q$-Bernoulli polynomials of order $r$ which are defined in [5, 6].

Let $\log x = \sum_{k\geqslant 1} (-1)^{k-1} \frac{(x-1)^k}{k}$ be the $p$-adic log function. This sum is convergent for $|x-1|_p < 1$. From now, we use the notation as follows:

$$L(y) = L(y_1, y_2, \ldots, y_r) = \sum_{j=1}^{r} y_j.$$

The function $G_{p,q}(L:x)$ generalizing the $q$-extension of $p$-adic log gamma function is defined by

$$G_{p,q}(L:x) = G_{p,q}(L(y) : x_1, x_2, \ldots, x_n) = \int_{\mathbb{Z}_p^{\times r}} \prod_{1\leqslant i\leqslant n} \log[x_i + L(y)] \, \mathrm{d}\mu_q(y)$$

where each $x_i \in \mathbb{C}_p^\times$.

**Remarks.**

(1) We call $G_{p,q}(L:x)$ the multiple $p$-adic $q$-log gamma function.

(2) Note that

$$\begin{aligned} G_{p,q}(L:x) = &\lim_{N_1,\ldots,N_r\to\infty} \frac{1}{\left[c_1 p^{N_1}\right] \cdots \left[c_r p^{N_r}\right]} \\ &\times \sum_{1\leqslant j\leqslant r} \sum_{0\leqslant m_j < c_j p^N}^{*} q^{m_1+\cdots+m_r} \log[x_i + L(m)] \end{aligned}$$

where $m = (m_1, \ldots, m_r) \in \mathbb{Z}_p^{\times r}$.

(3) For any integer $k$, if we define a $q$-Bernoulli number with order $k$ as

$$\left( \mathrm{e}^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{j+1}{[j+1]} (-1)^j \left( \frac{1}{1-q} \right)^j \frac{t^j}{j!} \right)^k = \sum_{n=0}^{\infty} \frac{\overline{\beta}_n^{(k)}}{n!} t^n$$

then, by Corollary 1, it is easy to see that

$$\bar{\beta}_n^{(k)} = \sum_{n=a_1+\cdots+a_k} \binom{n}{a_1, \ldots, a_k} \beta_{a_1}\beta_{a_2}\cdots\beta_{a_k}$$

where $\beta_{a_k}$ are the Carlitz $q$-Bernoulli numbers [5, 6].

**Theorem 2.** *Let $L_{p,q}(s_1, \ldots, s_n : r)$ be p-adic q-L-functions in n-variables and $G_{p,q}(L: x)$ be the q-extension of p-adic log multiple gamma function. Then we have*

(1)
$$\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_n} Z(1, 1, \ldots, 1 : q) = (-1)^n G_{p,q}(L : x)$$

*where $Z(s_1, \ldots, s_n : q) = \prod_{i=1}^n (s_i - 1) L_{p,q}(s_1, s_2, \ldots, s_n : r)$.*

(2)
$$\prod_{1\leqslant i\leqslant n} (-1)^{a_i-2} \frac{1}{(a_i-1)!q^{(a_i-1)L}} \frac{\partial^{a_1-1}}{\partial[x_1]^{a_1-1}} \cdots \frac{\partial^{a_n-1}}{\partial[x_n]^{a_n-1}} G_{p,q}(L : x) = L_{p,q}(a_1, a_2, \ldots, a_n : r),$$

*where each $a_i$ is a positive integer bigger than 2.*

The proof of the above theorem is not difficult (cf. [5, 6]) .

**Remark.** For $s \in \mathbb{C}$, $q \in \mathbb{C}$ with $|q| < 1$, define

$$\zeta_q^{(h,k)}(s) = \sum_{a_1,\ldots,a_k=0}^{\infty} \frac{q^{h(a_1+\cdots+a_k)}}{[a_1+\cdots+a_k]^s} + (q-1)\frac{1-s+h}{1-s} \times \sum_{a_1,\ldots,a_k=0}^{\infty} \frac{q^{h(a_1+\cdots+a_k)}}{[a_1+\cdots+a_k]^{s-1}}$$

where $h$, $k$ are positive integers.

Note that $\zeta_q^{(h,k)}(s)$ is an analytic continuation for $\Re(s) > 1$. If $k = 1$, $m \in \mathbb{N} = \{1, 2, \ldots\}$, then it was known in [6] that $\zeta_q^{(h,1)}(1-m) = -\frac{\beta_m^{(h,1)}}{m}$, where $\beta_m^{(h,1)}$ are $q$-Bernoulli numbers with order $h$ which are defined in [6]. Finally, we would like to suggest the following question:

**Question.** Is there an analogue of Bernoulli numbers which $\zeta_q^{(h,k)}(1-m)$ can be viewed as interpolating, in the same way that $\zeta_q^{(h,1)}(1-m)$ interpolates the $q$-Bernoulli number with order $h$?

References [7–9] are not cited above but are also important to the reader.

## Acknowledgment

This work was supported by Korea Research Foundation grant KRF-99-005-D00026.

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